# PROBLEM SET 4

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Prove the following statements regarding a Hilbert space $\mathcal{X}$.

**Problem 1.** *(Polarization identity) For any $x, y \in \mathcal{X}$,*

$$4\langle x, y\rangle = \|x+y\|^2 - \|x-y\|^2 + i\|x+iy\|^2 - i\|x-iy\|^2.$$

*Proof.* For any $x, y \in \mathcal{X}$, we have

$$\|x+y\|^2 = \langle x+y, x+y\rangle = \|x\|^2 + \langle x, y\rangle + \langle y, x\rangle + \|y\|^2 \tag{1}$$

$$\|x-y\|^2 = \langle x-y, x-y\rangle = \|x\|^2 - \langle x, y\rangle - \langle y, x\rangle + \|y\|^2 \tag{2}$$

$$\|x+y\|^2 - \|x-y\|^2 = 2\langle x, y\rangle + 2\langle y, x\rangle \quad \text{by subtracting (2) from (1)} \tag{3}$$

Replacing $y$ by $iy$ in (3) we get

$$\|x+iy\|^2 - \|x-iy\|^2 = 2\langle x, iy\rangle + 2\langle iy, x\rangle = -2i\langle x, y\rangle + 2i\langle y, x\rangle \tag{4}$$

$$i\|x+iy\|^2 - i\|x-iy\|^2 = 2\langle x, y\rangle - 2\langle y, x\rangle \tag{5}$$

Therefore by adding (3) and (5) we get

$$\|x+y\|^2 - \|x-y\|^2 + i\|x+iy\|^2 - i\|x-iy\|^2 = 4\langle x, y\rangle.$$

□

**Problem 2.** *If $E \subset \mathcal{X}$, $(E^\perp)^\perp$ is the smallest closed subspace of $\mathcal{X}$ containing $E$.*

*Proof.* Since perpendicular subspace of any subspace is closed, $(E^\perp)^\perp$ is closed. And it follows easily from definition that $E \subset (E^\perp)^\perp$. Moreover, since $(E^\perp)^\perp$ is closed, we see $\bar{E} \subset (E^\perp)^\perp$. Claim that $\bar{E} = (E^\perp)^\perp$, hence $(E^\perp)^\perp$ is the smallest closed subspace of $\mathcal{X}$ containing $E$. Indeed, otherwise there exists $x \in (E^\perp)^\perp \backslash \bar{E}$, then by Hahn-Banach theorem, there exists $f \in \mathcal{X}^*$ so that $f|_{\bar{E}} = 0$ and $f(x) \neq 0$. Then by theorem 5.25 there exists $y \in \mathcal{X}$ so that $f(-) = \langle -, y\rangle$. Therefore $y \in E^\perp$ and $\langle x, y\rangle \neq 0$, but this violates the assumption that $x \in (E^\perp)^\perp$. □

**Problem 3.** *Every closed convex set $K \subset \mathcal{X}$ has a unique element of minimal norm.*

*Proof.* The result is trivial if $0 \in K$, so we may assume $0 \in K$ and in particular if $x \in K$ then $-x \notin K$. Let $\delta = \inf\{\|x\| : x \in K\}$ and let $\{x_n\}$ be a sequence in $K$ such that $\|x_n\| \to \delta$. Since $K$ is convex, $\frac{1}{2}x_n + \frac{1}{2}x_m \in K$, so $\|\frac{1}{2}x_n + \frac{1}{2}x_m\| \geq \delta$. Therefore by parallelogram law,

$$\begin{aligned}\|x_n - x_m\|^2 &= 2(\|x_n\|^2 + \|x_m\|^2) - \|x_n + x_m\|^2 \\ &\leq 2(\|x_n\|^2 + \|x_m\|^2) - 4\delta^2 \to 0 \quad \text{as } n, m \to \infty.\end{aligned}$$

So $\{x_n\}$ is a Cauchy sequence, let $x = \lim x_n$. We see $x \in K$ since $K$ is closed, and $\|x\| = \lim \|x_n\| = \delta$ achieves the minimal norm.

To see the uniqueness of the norm minimizer, let $x, y \in K$ such that $\|x\| = \|y\| = \delta$. Consider the function

$$f(t) = \|tx + (1-t)y\|^2 = 2t(t-1)\cdot(\delta^2 - Re(\langle x, y\rangle)) + \delta^2, \quad t \in [0,1].$$

By Schwartz inequality, $Re(\langle x, y\rangle) \leq |\langle x, y\rangle| \leq \|x\|\|y\| = \delta^2$ where equality holds if and only if $x = \lambda y$ and $Re(\lambda) = |\lambda| = 1$, *i.e.* $x = y$. So if $x \neq y$, then $f(t)$ has a minimum at $\frac{1}{2}$ with

$$f(\frac{1}{2}) = \|\frac{1}{2}x + \frac{1}{2}y\|^2 = \frac{1}{2}\delta^2 + \frac{1}{2}Re(\langle x, y\rangle) < \delta^2.$$

But $\frac{1}{2}x + \frac{1}{2}y \in K$ by convexity and $\|\frac{1}{2}x + \frac{1}{2}y\| < \delta$, this contradicts the definition of $\delta$. Therefore $x = y$ as desired. □

**Problem 4.** *Let $\mathcal{X}$ be an infinite-dimensional Hilbert space.*

*(1) Every orthonormal sequence in $\mathcal{X}$ converges weakly to 0.*
*(2) The unit sphere $S = \{x : \|x\| = 1\}$ is weakly dense in the unit ball $B = \{x : \|x\| \leq 1\}$.*

*Proof.* (1) Let $\{x_n\}$ be an orthonormal sequence in $\mathcal{X}$ and $f \in \mathcal{X}^*$. By theorem 5.25, $f(-) = \langle -, y\rangle$ for some $y \in \mathcal{X}$. Then by Bessel's inequality, $\sum_{n=1}^{\infty} |\langle x_n, y\rangle|^2 \leq \|y\|^2 < \infty$, thus the series $\sum_{n=1}^{\infty} |\langle x_n, y\rangle|^2$ converges, therefore $f(x_n) = \langle x_n, y\rangle \to 0$ as $n \to \infty$. This proves $\{x_n\}$ weakly converges to 0.

(2) Let $x \in B$, we show $x$ is a weak limit of some sequence in $S$. This is trivial if $\|x\| = 1$, we may assume $\|x\| < 1$. Since $\mathcal{X}$ is infinite-dimensional, we can find an orthogonal sequence $\{x_n\}$ with $\langle x, x_n\rangle = 0$ and that (by a scaling if necessary) $\|x_n\|^2 = 1 - \|x\|^2$ for all $n$. We claim $\{x + x_n\} \subset S$ and converges to $x$ weakly. Indeed, $\|x + x_n\|^2 = \|x\|^2 + \|x_n\|^2 = 1$, and by (1), $\{x_n/(1 - \|x\|)\}$ weakly converges to 0, hence so is $\{x_n\}$, therefore $\{x + x_n\}$ converges weakly to 0.

□

**Problem 5.** *Let $A$ and $B$ be non-empty sets. Then $\ell^2(A)$ is isomorphic to $\ell^2(B)$ iff $A$ and $B$ have equal cardinality.*

*Proof.* On the one hand, if $A, B$ have the same cardinality, then let $\phi : A \to B$ be a bijection, then $\phi$ naturally induces a map $e_\alpha \mapsto e_{\phi(\alpha)}$ between the (canonical) set of orthonomal basis of $\ell^2(A)$ to that of $\ell^2(B)$, we may linearly extend this to a linear map $T_\phi : \ell^2(A) \to \ell^2(B)$. It is clear from the definition that $T_\phi$ is unitary (hence bounded). Moreover $T_\phi$ has a unitary inverse $T_{\phi^{-1}}$. So $T_\phi$ is a unitary isomorphism.

On the other hand, if $T : \ell^2(A) \to \ell^2(B)$ is an isomorphism, then $\ell^2(B)$ has two sets of base, namely $\{Te_\alpha\}_{\alpha\in A}$ and $\{e_\beta\}_{\beta\in B}$. Then the desired result follows from the following lemma. □

**Lemma.** *Let $V$ be a topological vector space, then any two sets of maximal independent vectors in $V$ have the same cardinality.*

*Proof of lemma.* We need help from our friend Zorn. Let $\{u_\alpha\}_{\alpha\in A}$ and $\{v_\beta\}_{\beta\in B}$ be two sets of maximal independent vectors. Consider the set $\mathcal{P}$ of pairs $(K, \phi)$ where $K$ is a subset of $A$ and $\phi : K \to B$ is an injection and $\{u_\alpha, v_\beta\}_{\alpha\in K, \beta\in B\setminus\phi(K)}$

is a maximal independent set. We equip $\mathcal{P}$ with a partial order $\leq$ by declaring $(K,\phi) \leq (L,\eta)$ if $K \subset L$ and $\eta|_K = \phi$.

Fix $\alpha_0 \in A$, by maximality $\{u_{\alpha_0}, v_\beta : \beta \in B\}$ is linear dependent, hence we may find a linear relation $\lambda_0 u_{\alpha_0} + \sum_{\beta\in B} \mu_\beta v_\beta = 0$. Notice $\lambda_0 \neq 0$ otherwise we have a relation among $\{v_\beta\}_{\beta\in B}$, also notice that not all $\mu_\beta$ are zero otherwise we have a relation among $\{u_\alpha\}_{\alpha\in A}$. Choose some $\beta_0 \in B$ so that $\mu_{\beta_0} \neq 0$. Then it is easy to show $\{u_{\alpha_0}, v_\beta : \beta \in B\backslash\{\beta_0\}\}$ is maximal linearly independent. This means, we have a member $(\{\alpha_0\}, \phi_0 : \alpha_0 \mapsto \beta_0) \in \mathcal{P}$, so $\mathcal{P}$ is non-empty.

By Zorn's lemma, $\mathcal{P}$ has a maximal element, say $(C,\phi)$. We claim $C = A$. Otherwise, there exists some $\alpha' \in A\backslash C$. Then by maximality of $\{u_\alpha, v_\beta\}_{\alpha\in C, \beta\notin\phi(C)}$, we see $\{u_{\alpha'}, u_\alpha, v_\beta : \alpha \in C, \beta \notin \phi(C)\}$ is dependent. Then a similar argument as before proves there exists some $\beta' \notin \phi(C)$ so that $\{u_{\alpha'}, u_\alpha, v_\beta : \alpha \in C, \beta \notin \phi(C), \beta \neq \beta'\}$ is maximal linearly independent. This means, we can extend $\phi : C \to B$ to be a new injection $\phi' : C \cup \{\alpha'\} \to B$ by mapping $\alpha'$ to $\beta'$. This contradicts maximality of $(C,\phi)$. So $C = A$, and thus we have an injection $A \to B$. Similarly, we can construct an injection $B \to A$. So finally $card(A) = card(B)$. □

**Problem 6.** *(The mean ergodic theorem) Let $U$ be an unitary operator on the Hilbert space $\mathcal{X}$, $\mathcal{M} = \{x : Ux = x\}$, $P$ the orthogonal projection onto $\mathcal{M}$, and $S_n = \frac{1}{n}\sum_0^{n-1} U^j$. Then $S_n \to P$ strongly.*

*Proof.* Before we go to prove $S_n \to P$, let's consider the subspace $\mathcal{N} := \{Ux - x | x \in \mathcal{X}\}$, we show $\overline{\mathcal{N}} = \mathcal{M}^\perp$. First we notice that for any $y \in \mathcal{M}$,

$$\langle Ux - x, y\rangle = \langle Ux, y\rangle - \langle x, y\rangle = \langle x, Uy\rangle - \langle x, y\rangle = \langle x, y\rangle - \langle x, y\rangle = 0,$$

therefore $\mathcal{N} \subset \mathcal{M}^\perp$, hence so is $\overline{\mathcal{N}}$. Suppose for contradiction that $\overline{\mathcal{N}} \neq \mathcal{M}^\perp$, then there exists $0 \neq v \in \mathcal{M}^\perp$ perpendicular to all vectors in $\mathcal{N}$, in particular $v$ is perpendicular to $Uv - v$ and $U^2v - Uv$. Therefore $\langle U^2v, v\rangle = \langle Uv, v\rangle = \|v\|^2$, hence

$$\begin{aligned}\|Uv - v\|^2 &= \langle Uv, Uv\rangle - \langle Uv, v\rangle - \langle v, Uv\rangle + \langle v, v\rangle \\ &= \langle U^2v, v\rangle - \langle Uv, v\rangle - \langle Uv, v\rangle + \langle v, v\rangle = 0.\end{aligned}$$

It follows that $Uv = v$, *i.e.* $v \in \mathcal{M}$. But by assumption $v \in \mathcal{M}^\perp$, so $v = 0$ which is a contradiction.

Now we are ready to show $S_n \to P$ strongly. Note that since $\mathcal{M}$ and $\mathcal{M}^\perp$ is an orthogonal (closed) decomposition of $\mathcal{X}$, it suffices to show $S_n|_{\mathcal{M}} \to P|_{\mathcal{M}}$ strongly and $S_n|_{\mathcal{M}^\perp} \to P|_{\mathcal{M}^\perp}$ strongly. Since both $S_n$ and $P$ are the identity map when restricted to $\mathcal{M}$, it remains to show $S_n|_{\mathcal{M}^\perp} \to P|_{\mathcal{M}^\perp} = 0$ strongly, but since $\overline{\mathcal{N}} = \mathcal{M}^\perp$ we only need to show $S_n|_{\mathcal{N}} \to 0$ strongly. Let $w = Ux - x \in \mathcal{N}$, then

$$\|S_n w\| = \|S_n(U - I)x\| = \|\frac{1}{n}(U^n - I)x\| \leq \frac{1}{n}(\|U^n\| + \|I\|)\|x\|,$$

where $I$ is the identity map. Since $U$ is unitary, so is $U^n$, thus $\|U^n\| \leq 1$. Therefore

$$\|S_n w\| \leq \frac{2}{n}\|x\| \to 0 \quad n \to \infty$$

for any $w \in \mathcal{N}$. This completes the proof. □

In the following problems $X$ is a locally compact Hausdorff space.

**Problem 7.** *Let $Y$ be a closed subset of $X$, and $\mu$ a Radon measure on $Y$. Prove that $I(f) = \int f|_Y d\mu$ is a positive linear functional on $C_c(X)$, and its induced Radon measure $\nu$ on $X$ is given by $\nu(E) = \mu(E \cap Y)$.*

*Proof.* Since integration is linear and preserves signs, it follows $I$ is a positive linear functional on $C_c(X)$. To proceed, we first notice that $Y^c$ is open and for any $f \prec Y^c$, $f|_Y \equiv 0$, so by Riesz representation theorem,

$$\nu(Y^c) = \sup\{\int_Y f|_Y d\mu : f \in C_c(X), f \prec Y^c\} = 0.$$

This implies $\nu(E) = \nu(E \cap Y)$ for any Borel set $E$. Then to show $\nu(E) = \mu(E \cap Y)$, it is a (somewhat tedious) routine to take the following steps.

*Step 1.*(compact sets) Let $K$ be a compact subset of $X$. Then by Riesz representation theorem,

$$\nu(K) = \inf\{\int_Y f|_Y d\mu : f \in C_c(X), f \geq \chi_K\}.$$

Since $Y$ is closed in $X$ and $X$ is locally compact Hausdorff, any compact set in $X$ intersecting $Y$ is compact in $Y$ and any compact set in $Y$ is compact in $X$. Notice $\mu$ is the unique Radon measure on $Y$ representing $\int_Y - d\mu$, then

$$\mu(K \cap Y) = \inf\{\int_Y g d\mu : g \in C_c(Y), g \geq \chi_{K\cap Y}\}.$$

Now let $f \in C_c(X)$ with $f \geq \chi_K$, then $f|_Y \in C_c(Y)$ and $f|_Y \geq \chi_{K\cap Y}$, so

$$\int_Y f|_Y d\mu \geq \inf\{\int_Y g d\mu : g \in C_c(Y), g \geq \chi_{K\cap Y}\} = \mu(K \cap Y).$$

Taking infimum on left hand side, we get $\nu(K) \geq \mu(K \cap Y)$. On the other hand,

$$\nu(K \cap Y) = \inf\{\int_Y f|_Y d\mu : f \in C_c(X), f \geq \chi_{K\cap Y}\}.$$

So if we take any $g \in C_c(Y)$ with $g \geq \chi_{K\cap Y}$, then $g \in C_c(X)$ as well, hence $\int_Y g d\mu \geq \mu(K \cap Y)$. Taking infimum on left side, we get $\mu(K \cap Y) \geq \nu(K \cap Y) = \nu(K)$. Thus we have $\nu(K) = \mu(K \cap Y)$ for all compact $K$.

*Step 2.* (open sets) Let $U$ be an open set of $X$. If $K \subset U$ and $K$ compact, then $K \cap Y \subset U \cap Y$, thus $\nu(K) = \mu(K \cap Y) \leq \mu(U \cap Y)$. Therefore, by inner regularity

$$\nu(U) = \sup\{\nu(K) : K \subset U, K \text{ compact}\} \leq \mu(U \cap Y).$$

On the other hand, if $K' \subset U \cap Y$ with $K'$ compact in $Y$, then $K' \subset U$ and $K'$ is compact in $X$, so

$$\mu(K') = \mu(K' \cap Y) = \nu(K') \leq \sup\{\nu(K) : K \subset U, K \text{ compact}\} = \nu(U).$$

Take supremum on left side, we get

$$\mu(U \cap Y) = \sup\{\mu(K') : K' \subset U \cap Y, K' \text{ compact in } Y\} \leq \nu(U).$$

This proves $\nu(U) = \mu(U \cap Y)$ for all open $U$.

*Step 3.*(Borel sets) Let $E$ be a Borel set. Let $U$ be open with $E \subset U$, then $\nu(U) = \mu(U \cap Y) \geq \mu(E \cap Y)$, hence by outer regularity,

$$\nu(E) = \inf\{\nu(U) : E \subset U, U \text{ open}\} \geq \mu(E \cap Y).$$

On the other hand, let $V$ be any open subset of $Y$ that contains $E \cap Y$, then $V = U \cap Y$ for some open subset $U$ of $X$. We may assume, replacing $U$ by $U \cup Y^c$ if necessary, that $E \subset U$. Then $\mu(V) = \mu(U \cap Y) = \nu(U) \geq \nu(E)$. Taking infimum, we have

$$\mu(E \cap Y) = \inf\{\mu(V) : E \cap Y \subset V, V \text{ open in } Y\} \geq \nu(E).$$

This (finally) completes the proof! ☐

**Problem 8.** *Let $\mu$ be a Radon measure on $X$. Let $N$ be the union of all open $U \subset X$ such that $\mu(U) = 0$. Then $N$ is open, $\mu(N) = 0$, and if $V$ is open and $V \setminus N \neq \emptyset$ then $\mu(V) > 0$. $N^c$ is called the* support *of $\mu$. Prove $x \in \operatorname{supp}(\mu)$ iff $\int f d\mu > 0$ for every $f \in C_c(X, [0,1])$ s.t $f(x) > 0$.*

*Proof.* It is clear that $N$ is open since $N$ is a union of open sets. To see $\mu(N) = 0$, we use inner regularity. Let $K$ be any compact set contained in $N$, then $K$ is covered by all those open $U$ with $\mu(U) = 0$. By compactness, there is a finite subcover, hence $\mu(K)$ is no bigger than the sum of measures of those finite open sets that cover $K$, which is zero. It follows $\mu(K) = 0$ for all $K$ compact contained in $N$, thus by inner regularity $\mu(N) = 0$. Moreover, if $V$ is open with $V \backslash N \neq \varnothing$, then $\mu(V) > 0$, since otherwise $V \subset N$ by definition and $V \backslash N = \varnothing$.

Now if $x \in \operatorname{supp}(\mu)$, then for every $f \in C_c(X, [0,1])$ with $f(x) > 0$, there is an open neighborhood $V$ of $x$ such that $f|_V \geq f(x)/2 > 0$. Since $x \in V \cap N^c$, $x \in V \backslash N \neq \varnothing$, therefore

$$\int_X f d\mu \geq \int_V f d\mu \geq f(x)\mu(V)/2 > 0.$$

Conversely if $x \notin \operatorname{supp}(\mu)$, then $x \in N$. Since $X$ is locally compact, there exists compact $K$ with $x \in K \subset N$. By locally compact version of Urysohn's lemma, there exists $f \in C(X, [0,1])$ such that $f = 1$ on $K$ and $f = 0$ outside a compact subset of $U$. Then $f(x) = 1 > 0$ but

$$\int_X f d\mu = \int_N f d\mu \leq \mu(N) = 0.$$

☐